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A Balloon Net Discovering Improved Solutions in an Unsolved Geometric Problem

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ABSTRACT

A balloon net model is introduced and demonstrated for discovering improved solutions in one of unsolved problems in geometry which is referred to as a problem of "Spreading points in a square". How should n points be arranged in a unit square so that the minimum distance between them is the greatest? Note that $d(n)$ is the greatest possible minimum distance between n points in a unit square. Exact results are known for $n \leq 9$ and $n = 14, 16, 25$, and 36. Many investigators including Schaer, Meir, Kirchner, Wengerodt, Goldberg, Schluter, Valette and others have studied this geometrical problem [1]–[7] for many years. The best known result is summarized in the book of "Unsolved Problems in Geometry"(H.T.Croft, K.J.Falconer and R.K.Guy/1991) [1]. We have found improved solutions for $n = 13$ and $n = 15$.

1. Introduction

The proposed balloon net model in this paper gives improved configurations of the problem: spreading points in a unit squre. This problem is one of unsolved problems in geometry. This problem is how should n points be arranged in an 1×1 unit square so that the minimum distance between them is the greatest. The aim of this problem is:

- Locate n points in 1×1 unit
- maximize the minimum distance between points

In other words, this means what is the maximum diameter of n equal circles that can be packed into a $(1 + d_n) \times (1 + d_n)$ unit square (where d_n is the maximum distans for n points). Some results ($n \leq 9$ and 14, $n = i^2$) are already known. Fig. 1 shows one of the known result of this problem($n = 7$).

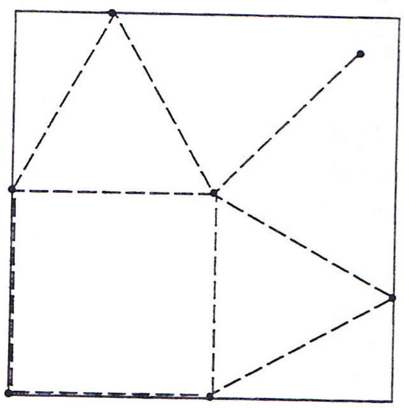


Fig. 1: Known result for n=7(Schaer and Meir/1965)

The proposed balloon net model gives new solution for $n = 13$ and $n = 15$ over the best known solutions.

2. Balloon net model and vector neuron

The balloon net model has evolved from the artificial neural network model where the motion equation of the nonlinear dynamic neural system has been used for solving optimization and combinatorics problems [8]. The motion equation represents the entire connectivities of the artificial neural network system. The proposed balloon net model is composed of n motion equations for n spreading points problem. There are two differnces between proposed balloon net model and normaly neural network model: One is that the input/output function of a spreading neuron in the artificial neural network is nonlinear while that in the balloon net model is a linear function. The other is that proposed model using a vector neuron, which has a vector value for each points. Each vector neuron has two values of floating point for X and Y axis.

In the balloon net model the followings are considered:

1. N tiny balloons(circles) are generated and arranged initially around the center of a unit square.
2. N ballooons gradually grow with time elapse by updating the coordinates of n balloons. When two or more than two balloons collide with each other, they will slowly bounce back. Bouncing forces gradually decrease with time elapse.
3. The state of n balloons reach the equilibrium state when n balloons have no room to grow any further.

For solving the n spreading points problem, we use n circles instead of n points and a $(1+d_n)\times(1+d_n)$ unit square instead of 1×1 unit square.

Fig. 2 shows behaviors of the proposed balloon net model.

The center coordinate of the ith balloon is denoted by $\vec{P_i} = P_i(X_i, Y_i)$ for $i = 1, ..., n$. The motion equation of the ith balloon used for representing bouncing forces against other balloons is given by:

$$\frac{d\vec{P_i}}{dt} = g\left(n, d_n(t), t\right) \sum_{j=1, j\neq i}^{N} f\left(\vec{P_i}, \vec{P_j}\right) \qquad (1)$$

where $f\left(\vec{P_i}, \vec{P_j}\right) = \vec{P_i} - \vec{P_j}$ if P_j is the coordinate of the jth balloon and the jth balloon must be the nearest balloon to the ith balloon, and $\vec{0}$ otherwise. Note that $(n, d_n(t), t)$ is a bouncing force function: $g\left(n, d_n(t), t\right) = A\frac{\sqrt{n}}{d_n(t)t}$ where A is a constant coefficient and $d_n(t)$ is the minimum distance between points in time t. The goal of this problem is to minimize the computational energy function E. The energy function E is described by:

$$E = \sum \frac{d\vec{P_i}}{dt} \qquad (2)$$

The coordinate of $\vec{P_i}$ is updated by the first order Euler method:

$$\vec{P_i}(t+1) = \vec{P_i}(t) + \Delta\vec{P_i}(t) \qquad (3)$$

where $\Delta\vec{P_i}(t)$ is gived by Eq.1.

The proposed algorithm is summarized by the following steps:

step1. Initialize n balloons' randomized coordinates around the center of a unit square.

step2. Use the motion equation in Eq.1 for $i = 1, ..., n$.

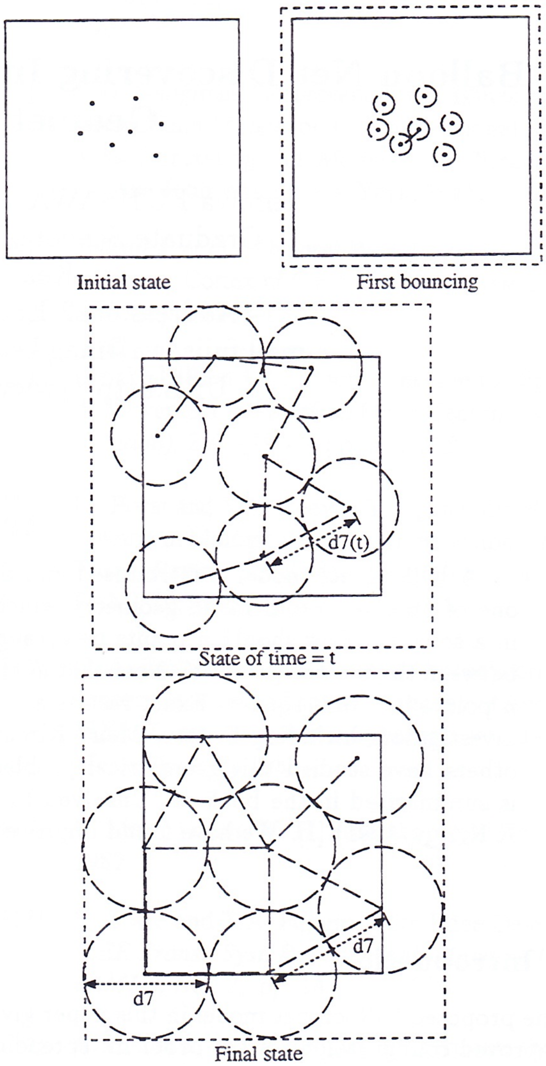


Fig. 2: Behaviors of balloons($n = 13$)

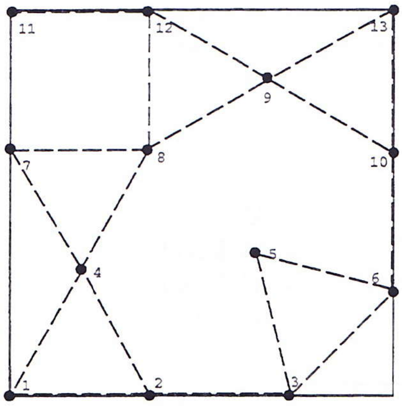
step3. Update n ballons' coordinates by Eq.3 for $i = 1, ..., n$.

step4. Terminate this procedure if energy function $E \simeq 0$(Eq.2).

step5. Go to step-2.

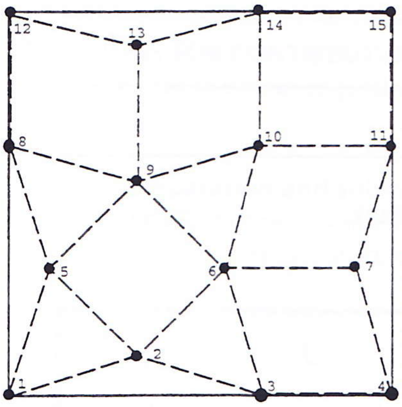
3. Results

The balloon net model system has been simulated on a sparc-10 workstation. The proposed system has discovered improved configurations for $n = 13$ and $n = 15$ where the n spreading points problem is composed of n motion equations. Fig. 3 shows our solution $d_{13} = 0.366093$ for $n = 13$ which is better than the best existing solution $d_{13} = \frac{\sqrt{3}-1}{2} \simeq 0.366024$ proposed by Schluter [6] where both have the similar configuration. Fig. 4(a) shows our simulated solution for $n = 15$ which is the significantly different configuration $d_{15} = 0.341024$ from the best existing solution [5] with $d_{15} = \frac{4}{8+\sqrt{2}+\sqrt{6}} \simeq 0.337162$. We can easily predict a better solution $d_{15} = \frac{1+\sqrt{2}-\sqrt{3}}{2} \simeq 0.341081$ in Fig. 4(b). Our system can generate a significantly improved solution for $n = 15$ in a few minutes on a regular workstation.



d13=0.366093
1(0.0000000,0.0000000)
2(0.3664192,0.0000000)
3(0.7325145,0.0000000)
4(0.1832125,0.3169533)
5(0.6342284,0.3649280)
6(1.0000000,0.2499525)
7(0.0000000,0.6339038)
8(0.3661633,0.6340544)
9(0.6882784,0.8080264)
10(1.0000000,0.6160514)
11(0.0000000,1.0000000)
12(0.3765559,1.0000000)
13(1.0000000,1.0000000)

Fig. 3: Our solution for $n = 13$



d15=0.341024
1(0.0000000,0.0000000)
2(0.3295045,0.0880553)
3(0.6589679,0.0000000)
4(1.0000000,0.0000000)
5(0.0883947,0.3293971)
6(0.5705582,0.3293644)
7(0.9116082,0.3293692)
8(0.0000000,0.6587947)
9(0.3294418,0.5706562)
10(0.6589698,0.6587832)
11(1.0000000,0.6588786)
12(0.0000000,1.0000000)
13(0.3295803,0.9116862)
14(0.6589758,1.0000000)
15(1.0000000,1.0000000)

(a)

$$d15 = \frac{1+\sqrt{2}-\sqrt{3}}{2}$$

1$(0,0)$
2$(\frac{1-\sqrt{2}+\sqrt{3}}{4}, \frac{-1-2\sqrt{2}+\sqrt{3}+\sqrt{6}}{4})$
3$(\frac{1-\sqrt{2}+\sqrt{3}}{2}, 0)$
4$(1,0)$
5$(\frac{-1-2\sqrt{2}+\sqrt{3}+\sqrt{6}}{4}, \frac{1-\sqrt{2}+\sqrt{3}}{4})$
6$(\frac{3+\sqrt{3}-\sqrt{6}}{4}, \frac{1-\sqrt{2}+\sqrt{3}}{4})$
7$(\frac{5+2\sqrt{2}-\sqrt{3}-\sqrt{6}}{4}, \frac{1-\sqrt{2}+\sqrt{3}}{4})$
8$(0, \frac{1-\sqrt{2}+\sqrt{3}}{2})$
9$(\frac{1-\sqrt{2}+\sqrt{3}}{4}, \frac{3+\sqrt{3}-\sqrt{6}}{4})$
10$(\frac{1-\sqrt{2}+\sqrt{3}}{2}, \frac{1-\sqrt{2}+\sqrt{3}}{2})$
11$(1, \frac{1-\sqrt{2}+\sqrt{3}}{2})$
12$(0,1)$
13$(\frac{1-\sqrt{2}+\sqrt{3}}{4}, \frac{5+2\sqrt{2}-\sqrt{3}-\sqrt{6}}{4})$
14$(\frac{1-\sqrt{2}+\sqrt{3}}{2}, 1)$
15$(1,1)$

(b)

Fig. 4: Our solution for $n = 15$(a) and predicted value(b)

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